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Free flexural vibration analysis of symmetric honeycomb panels

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Abstract

The free flexural vibration of symmetric rectangular honeycomb panels having simple support boundary conditions is investigated in this paper using the classical plate theory, Mindlin's improved plate theory, and Reddy's third-order plate theory. The honeycomb core of hexagonal cells is modeled as a thick layer of orthotropic material whose physical and mechanical properties are determined using the Gibson and Ashby correlations. The comparative studies conducted on aluminum honeycomb panels indicate that both the classical and improved plate theories are inadequate for the flexural vibration of honeycomb panels.

1. Introduction

Honeycomb sandwich panels are commonly used in various industries because of their high strength-to-weight ratios, desirable acoustic properties, and many other advantages. It is well understood that the classical plate theory (CPT) yields unacceptable results for composite plates because the transverse shear deformations are neglected. The first-order shear deformation plate theory (FSDPT), developed by Mindlin [1] for isotropic plates, and Yang et al. [2] for laminated

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Nomenclature

A_{ij}	elements of extensional (first order) stiffness matrix
a_{ij}	A_{ij} normalized with respect to E_0h^3
a	panel/plate dimension in the x direction
a_k	lower thickness coordinate of layer k
b	panel/plate dimension in the y direction
b_k	upper thickness coordinate of layer k
D_{ij}	elements of flexural (third order) stiffness matrix
d_{ij}	D_{ij} normalized with respect to E_0h^3
E_1, E_2	moduli of elasticity
E_0	reference modulus of elasticity
F_{ij}	elements of fifth order stiffness matrix
f_{ij}	F_{ij} normalized with respect to E_0h^5
G_{ij}	shear moduli
H_{ij}	elements of seventh order stiffness matrix
h	plate/panel thickness
h_{ij}	H_{ij} normalized with respect to E_0h^7
h_c	core thickness
I_k	kth order mass moment of inertia
i_k	I_k normalized with respect to ρ_0h^k
m	integer representing number of half sine waves in the x direction
$\overline{m}$	$= m\pi\phi_a$
n	integer representing number of half sine waves in the y direction
$\overline{n}$	$= n\pi\phi_b$
$Q_{ij}^{(k)}$	elements of material property matrix for layer k; $k = 1$ or f (faces); $k = 2$ or c (core)
t	time
u_x, u_y, u_z	displacements in the x, y, and z directions, respectively
w	lateral displacement
W	lateral displacement normalized with respect to plate/panel thickness
x, y, z	Cartesian coordinates

Greek symbols

α	core-to-panel thickness ratio
ε_{ij}	strain components
ϕ_a	thickness to length ratio $= h/a$
ϕ_b	thickness to width ratio $= h/b$
η	non-dimensional coordinate ($= y/b$)
λ^2	eigenvalue
ν_{ij}	Poisson's ratios
ρ	density
ρ_0	reference density
$\rho^{(k)}$	density of layer k material
σ_{ij}	stress components
ω	natural frequency
ξ	non-dimensional coordinate ($= x/a$)
ψ_x	angle of rotation of material line normal to the mid-plane in the xoz plane
Ψ_x	amplitude of ψ_x
ψ_y	angle of rotation of material line normal to the mid-plane in the yoz plane
Ψ_y	amplitude of ψ_y

plates, yields acceptable results for thin and moderately thick plates. Reddy [3] developed a simplified third-order shear deformation plate theory (TSDPT). Compared to the FSDPT, the TSDPT, free from use of any shear correction factor, contains no additional field variables while satisfying the zero transverse shear stress conditions at the top and bottom surfaces of a plate. The TSDPT provides a parabolic distribution of the transverse shear stresses in the thickness direction. There are various other higher order plate theories available in the literature, e.g., Lo et al. [4]. Those higher order theories involve additional field variables, and are very complicated to use.

Honeycomb panels are commonly used in the aerospace industry, race car industry, and other industries where high strength-to-weight ratios and desirable acoustic properties are essential. A typical honeycomb sandwich panel of length a, width b and height h, shown in Fig. 1, consists of two face sheets and arrays of open cells glued to the inner surfaces of the face sheets. To achieve high flexural strengths or flexural natural frequencies, the honeycomb core height is usually about 80–95% of the total panel thickness.

A review of literature indicates that little work on flexural vibration of honeycomb panels has been carried out. Millar [5] investigated the behavior of honeycomb panels under acoustic load. Cunningham and White [6], and Cunningham et al. [7] studied free vibration of honeycomb panels. For an efficient analysis of free vibration and buckling of sandwich panels having porous cores, the equivalent continuum approach is often used in Ref. [8]. Lok and Cheng [9] considered a truss-core sandwich panel as a homogenous single layer orthotropic thick plate; they concluded that the equivalent continuum approach yields satisfactory results if the panel dimensions are significantly larger than the cell dimensions.

Experimental results and three-dimensional modeling conducted by Lai [10] have indicated that the honeycomb core can be modeled as an equivalent homogeneous orthotropic material in flexural vibration analyses. However, to use the continuum model for analyzing the free vibration of honeycomb panels, it is important that reliable values of the equivalent material properties of honeycomb cores be used. The equivalent material properties of honeycomb cores may be obtained from experiments, empirical correlations, and solid mechanics modeling. According to Bitzer [11], the material properties of primary importance are the out-of-plane properties; the in-plane properties are secondary. Wilfried reported in Ref. [12] that these properties are dependent not only on cell material and cell geometry, but also on the panel dimensions. Studies of honeycomb properties may be found in Refs. [13–15].

In this paper, the porous honeycomb core is considered as a homogenous orthotropic material whose equivalent material properties are determined using the correlations by Gibson and Ashby [13]. The three plate theories CPT, FSDPT and TSDPT are employed to analyze free flexural vibration of symmetric honeycomb panels. To obtain an exact analytical solution, the two in-plane material principal directions of honeycomb cores, x_1 and x_2, as shown in Fig. 2, are assumed to be the same as the panel in-plane coordinates, x and y, respectively. As a test case, numerical results were also obtained for isotropic plates.

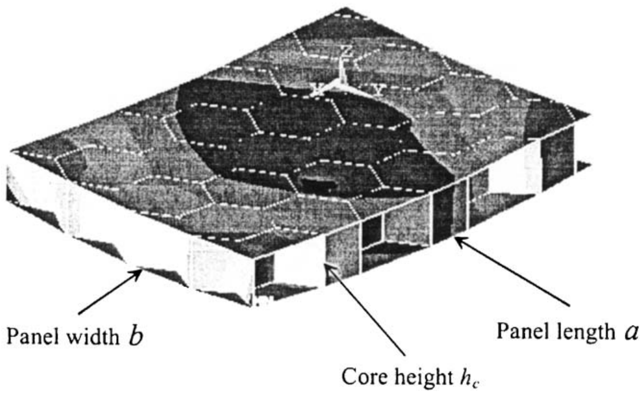


Fig. 1. A typical honeycomb panel.

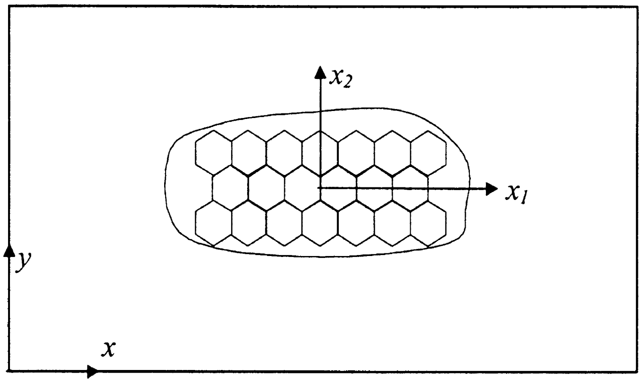


Fig. 2. A rectangular honeycomb panel having a special core alignment.

2. Three plate theories

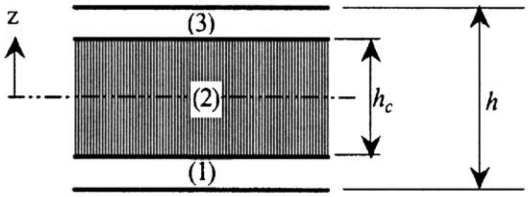
In this section, the governing differential equations for the small flexural vibration of symmetric honeycomb panels in terms of the displacements will be presented for the CPT, the IPT, and the TSDPT. Because bending and in-plane stretching/compression are uncoupled for symmetric honeycomb panels, the in-plane displacements not caused by bending are discarded in this paper. Since the normal stress and strain in the thickness direction of a plate are not included in any of the three plate theories, the following reduced stress–strain relationship in an isotropic or specially orthotropic constitutive layer may be used to formulating the governing equations of motion

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 & 0 & 0 \\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & Q_{44}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & Q_{55}^{(k)} & 0 \\ 0 & 0 & 0 & 0 & Q_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix}. \quad (1)$$

The stiffness and inertia parameters may be established by considering a honeycomb panel as a three layered laminate structure. For a honeycomb panel, the bottom and top face sheets are designated as layers 1 and 3, respectively; the core is designated as layer 2. As shown in Fig. 3, each layer is bounded by the coordinates, a_k and b_k, in the thickness direction. These parameters appeared in the three plate theories as follows:

$$(A_{ij}, D_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{3} \int_{a_k}^{b_k} Q_{ij}^{(k)}(1, z^2, z^4, z^6)\,\mathrm{d}z \quad (i,j = 1,2,6;4,5),$$

$$(I_1, I_3, I_5, I_7) = \sum_{k=1}^{3} \int_{a_k}^{b_k} \rho^{(k)}(1, z^2, z^4, z^6)\,\mathrm{d}z. \quad (2)$$



Layers	a_k	b_k
1	$-h/2$	$-h_c/2$
2	$-h_c/2$	$h_c/2$
3	$h_c/2$	$h/2$

Fig. 3. Coordinates in the thickness direction for the core and two face sheets.

It is mentioned that in the IPT, the extensional stiffness parameters A_{44} and A_{55} are multiplied by a shear correctional factor ($\kappa = 0.8601$). For convenience, the stiffness and inertial parameters are non-dimensionalized in the following manner:

$$(a_{ij}, d_{ij}, f_{ij}, h_{ij}) = \left(\frac{A_{ij}}{E_0 h}, \frac{D_{ij}}{E_0 h^3}, \frac{F_{ij}}{E_0 h^5}, \frac{H_{ij}}{E_0 h^7}\right), \quad (i_1, i_3, i_5, i_7) = \left(\frac{I_1}{\rho_0 h}, \frac{I_3}{\rho_0 h^3}, \frac{I_5}{\rho_0 h^5}, \frac{I_7}{\rho_0 h^7}\right). \tag{3}$$

With the help of the core-to-panel thickness ratio, or $\alpha = h_c/h$, the non-dimensional stiffness and inertial parameters may be readily obtained for a symmetric honeycomb panel. They are written as

$$a_{ij} = \frac{Q_{ij}^{(c)}}{E_0}\alpha + \frac{Q_{ij}^{(f)}}{E_0}(1-\alpha), \quad d_{ij} = \frac{1}{12}\left[\frac{Q_{ij}^{(c)}}{E_0}\alpha^3 + \frac{Q_{ij}^{(f)}}{E_0}(1-\alpha^3)\right],$$

$$f_{ij} = \frac{1}{80}\left[\frac{Q_{ij}^{(c)}}{E_0}\alpha^5 + \frac{Q_{ij}^{(f)}}{E_0}(1-\alpha^5)\right], \quad h_{ij} = \frac{1}{448}\left[\frac{Q_{ij}^{(c)}}{E_0}\alpha^7 + \frac{Q_{ij}^{(f)}}{E_0}(1-\alpha^7)\right],$$

$$i_1 = \frac{\rho^c}{\rho_0}\alpha + \frac{\rho^{(f)}}{\rho_0}(1-\alpha), \quad i_3 = \frac{1}{12}\left[\frac{\rho^{(c)}}{\rho_0}\alpha^3 + \frac{\rho^{(f)}}{\rho_0}(1-\alpha^3)\right],$$

$$i_5 = \frac{1}{80}\left[\frac{\rho^{(c)}}{\rho_0}\alpha^5 + \frac{\rho^{(f)}}{\rho_0}(1-\alpha^5)\right], \quad i_7 = \frac{1}{448}\left[\frac{\rho^{(c)}}{\rho_0}\alpha^7 + \frac{\rho^{(f)}}{\rho_0}(1-\alpha^7)\right],$$

where E_0 is the reference modulus of elasticity; h is the total thickness of a honeycomb panel; h_c is the height of the honeycomb core; ρ_0 is the reference density used to normalize inertial quantities. For simplicity, the reference density and modulus of elasticity are chosen to be those of the face material in this paper.

2.1. Classical plate theory

In the classical plate theory, it is assumed that straight material lines normal to the plate mid-plane before deformation remain straight and normal to the mid-plane after deformation. The displacements of a material point (x, y, z) caused by bending may be expressed in terms of a single lateral displacement

$$u_x = -z\frac{\partial w(x,y,t)}{\partial x}, \quad u_y = -z\frac{\partial w(x,y,t)}{\partial y}, \quad u_z = w(x,y,t). \tag{4}$$

The equation of motion and the consistent boundary conditions for the lateral displacement, w, may be derived from the Hamilton's principle. The governing equation of free flexural of a symmetric honeycomb panel may be written as

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + I_1\ddot{w} - I_3\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right) = 0. \tag{5}$$

Introducing the transformations

$$x = \alpha\xi, \quad y = b\eta, \quad w(x,y,z) = hW(\xi,\eta)\sin(\omega t), \tag{6}$$

Eq. (5) may be normalized and written as

$$d_{11}\phi_a^4\frac{\partial^4 W}{\partial \xi^4} + 2(d_{12}+2d_{66})\phi_a^2\phi_b^2\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + d_{22}\phi_b^4\frac{\partial^4 W}{\partial \eta^4} - \lambda^4\left[i_1 W - i_3\left(\frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial \eta^2}\right)\right] = 0, \tag{7}$$

where $\phi_a = h/a$, $\phi_b = h/b$ and $\lambda^2 = \omega h\sqrt{\rho_0/E_0}$.

For a rectangular honeycomb panel simply supported along all four edges, an exact solution for the dimensionless amplitude of flexural vibration may be written as

$$W(x,y) = \sum_{m,n=1}^{\infty} Z_{mn}\sin m\pi\xi \sin n\pi\eta. \tag{8}$$

Substituting the above solution into Eq. (7), one may obtain the eigenvalue

$$\lambda^2 = \frac{\sqrt{d_{11}\overline{m}^4 + 2(d_{12}+d_{66})\overline{m}^2\overline{n}^2 + d_{22}\overline{n}^4}}{i_1 + i_3[\overline{m}^2 + \overline{n}^2]} \quad (m,n = 1,2,\ldots), \tag{9}$$

where $\overline{m} = m\pi\phi_a$, $\overline{n} = n\pi\phi_b$.

Solutions for symmetric honeycomb panels having other combinations of boundary conditions along each of the four edges may be obtained using the method of superposition, developed by Gorman [16] for the free vibration analysis of isotropic thin plates.

2.2. Improved plate theory

In the improved plate theory, it is assumed that straight material lines normal to the plate mid-plane before deformation remain straight after deformation. The displacements of a material point (x,y,z) caused by bending may be expressed in terms of the lateral displacement and two angles of rotation as

$$u_x = z\psi_x(x,y,t), \quad u_y = z\psi_y(x,y,t), \quad u_z = w(x,y,t). \tag{10}$$

The equation of motion for free flexural vibration of a symmetric honeycomb panel may be written as

$$\begin{bmatrix} A_{55}\frac{\partial^2}{\partial x^2}+A_{44}\frac{\partial^2}{\partial y^2} & A_{55}\frac{\partial}{\partial x} & A_{44}\frac{\partial}{\partial y} \\ -A_{55}\frac{\partial}{\partial x} & D_{11}\frac{\partial^2}{\partial x^2}+D_{66}\frac{\partial^2}{\partial y^2}-A_{55} & (D_{12}+D_{66})\frac{\partial^2}{\partial x\partial y} \\ -A_{44}\frac{\partial}{\partial y} & (D_{12}+D_{66})\frac{\partial^2}{\partial x\partial y} & D_{66}\frac{\partial^2}{\partial x^2}+D_{22}\frac{\partial^2}{\partial y^2}-A_{44} \end{bmatrix}\begin{Bmatrix} w \\ \psi_x \\ \psi_y \end{Bmatrix}$$
$$= -\begin{Bmatrix} I_1\ddot{w} \\ I_3\ddot{\psi}_x \\ I_3\ddot{\psi}_y \end{Bmatrix}. \tag{11}$$

Introducing the transformations

$$w = hW(\xi,\eta)\sin(\omega t), \quad \psi_x = \Psi_x(\xi,\eta)\sin(\omega t), \quad \psi_y = \Psi_y(\xi,\eta)\sin(\omega t). \tag{12}$$

Eq. (11) may be rewritten as

$$\begin{bmatrix} a_{55}\phi_a^2\frac{\partial^2}{\partial \xi^2}+a_{44}\phi_b^2\frac{\partial^2}{\partial \eta^2} & a_{55}\phi_a\frac{\partial}{\partial \xi} & a_{44}\phi_b\frac{\partial}{\partial \eta} \\ -a_{55}\phi_a\frac{\partial}{\partial \xi} & d_{11}\phi_a^2\frac{\partial^2}{\partial \xi^2}+d_{66}\phi_b^2\frac{\partial^2}{\partial \eta^2}-a_{55} & (d_{12}+d_{66})\phi_a\phi_b\frac{\partial^2}{\partial \xi\partial \eta} \\ -a_{44}\phi_b\frac{\partial}{\partial \eta} & (d_{12}+d_{66})\phi_a\phi_b\frac{\partial^2}{\partial \xi\partial \eta} & d_{66}\phi_a^2\frac{\partial^2}{\partial \xi^2}+d_{22}\phi_b^2\frac{\partial^2}{\partial \eta^2}-a_{44} \end{bmatrix}\begin{Bmatrix} W \\ \Psi_x \\ \Psi_y \end{Bmatrix}$$
$$+\lambda^4\begin{bmatrix} i_1 & 0 & 0 \\ 0 & i_3 & 0 \\ 0 & 0 & i_3 \end{bmatrix}\begin{Bmatrix} W \\ \Psi_x \\ \Psi_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{13}$$

The boundary conditions, which are consistent with the improved plate theory, are studied by Reismann in Ref. [17]. For a simply supported rectangular honeycomb panel, an exact solution satisfying boundary conditions along the four edges may be sought employing the series

$$W(\xi,\eta) = \sum_{m,n=1}^{\infty} Z_{mn}\sin m\pi\xi\,\sin n\pi\eta, \quad \Psi_x(\xi,\eta) = \sum_{m,n=1}^{\infty} X_{mn}\cos m\pi\xi\,\sin n\pi\eta,$$
$$\Psi_y(\xi,\eta) = \sum_{m,n=1}^{\infty} Y_{mn}\sin m\pi\xi\,\cos n\pi\eta. \tag{14}$$

Substituting Eq. (14) into Eq. (13), one obtains the following homogeneous algebraic equations, leading to a standard eigenvalue problem for free vibration:

$$[[M_{mn}]^{-1}[K_{mn}] - \lambda^4[I]]\begin{Bmatrix} Z_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{15}$$

where

$$[M_{mn}] = \begin{bmatrix} i_1 & 0 & 0 \\ 0 & i_3 & 0 \\ 0 & 0 & i_3 \end{bmatrix}, \quad [K_{mn}] = \begin{bmatrix} a_{55}\overline{m}^2 + a_{44}\overline{n}^2 & a_{55}\overline{m} & a_{44}\overline{n} \\ a_{55}\overline{m} & d_{11}\overline{m}^2 + d_{66}\overline{n}^2 + a_{55} & (d_{12}+d_{66})\overline{m}\,\overline{n} \\ a_{44}\overline{n} & (d_{12}+d_{66})\overline{m}\,\overline{n} & d_{66}\overline{m}^2 + d_{22}\overline{n}^2 + a_{44} \end{bmatrix}.$$

2.3. Third-order shear deformation plate theory

In the third-order shear deformation plate theory developed by Reddy [3,18], straight material lines normal to the plate mid-plane before deformation will no longer remain straight. The displacements of a material point (x, y, z) caused by bending may be written as

$$\begin{aligned} u_x &= \left(z - \frac{4z^3}{3h^2}\right)\psi_x(x,y,t) - \frac{4z^3}{3h^2}\frac{\partial w(x,y,t)}{\partial x}, \\ u_y &= \left(z - \frac{4z^3}{3h^2}\right)\psi_y(x,y,t) - \frac{4z^3}{3h^2}\frac{\partial w(x,y,t)}{\partial y}, \\ u_z &= w(x,y,t). \end{aligned} \tag{16}$$

Although Reddy's plate theory involves only three field variables, which are dependent on x, y and t, this theory allows transverse shear stresses/strains to vary in a quadratic polynomial, and in-plane normal stresses/strains and in-plane shear stress/strain in a cubic polynomial in the plate thickness direction. These features are important in dealing with honeycomb panels having thick and soft cores. The governing equations for free vibration of a symmetric honeycomb panel may be written as

$$\begin{aligned} &\overline{A}_{55}\frac{\partial^2 w}{\partial x^2} + \overline{A}_{44}\frac{\partial^2 w}{\partial y^2} - \frac{16}{9h^4}\left(H_{11}\frac{\partial^4 w}{\partial x^4} + 2(H_{12}+H_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + H_{22}\frac{\partial^4 w}{\partial y^4}\right) \\ &\quad + \frac{\partial}{\partial x}\left[\overline{A}_{55}\psi_x + \frac{4}{3h^2}\left(\overline{F}_{11}\frac{\partial^2 \psi_x}{\partial x^2} + (\overline{F}_{12} + 2\overline{F}_{66})\frac{\partial^2 \psi_x}{\partial y^2}\right)\right] \\ &\quad + \frac{\partial}{\partial y}\left[\overline{A}_{44}\psi_y + \frac{4}{3h^2}\left((\overline{F}_{12} + 2\overline{F}_{66})\frac{\partial^2 \psi_y}{\partial x^2} + \overline{F}_{22}\frac{\partial^2 \psi_y}{\partial y^2}\right)\right] \\ &= I_1\ddot{w} - \frac{16}{9h^4}I_7\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right) + \frac{4}{3h^2}\left(I_5 - \frac{4}{3h^2}I_7\right)\left(\frac{\partial \ddot{\psi}_x}{\partial x} + \frac{\partial \ddot{\psi}_y}{\partial y}\right), \end{aligned} \tag{17}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\left[A_{55}w+\frac{4}{3h^2}\left(\overline{F}_{11}\,\frac{\partial^2 w}{\partial x^2}+(\overline{F}_{12}+2\overline{F}_{66})\frac{\partial^2 w}{\partial y^2}\right)\right]\\
&\quad+\overline{D}_{11}\,\frac{\partial^2\psi_x}{\partial x^2}+\overline{D}_{66}\,\frac{\partial^2\psi_x}{\partial y^2}-\overline{A}_{55}\psi_x+(\overline{D}_{12}+\overline{D}_{66})\frac{\partial^2\psi_y}{\partial x\partial y}\\
&\quad=-\frac{4}{3h^2}\left(I_5-\frac{4}{3h^2}\,I_7\right)\frac{\partial\ddot{w}}{\partial x}+\left(I_3-\frac{8}{3h^2}\,I_5+\frac{16}{9h^4}I_7\right)\ddot{\psi}_x,
\end{aligned}\tag{18}
$$

$$
\begin{aligned}
&-\frac{\partial}{\partial y}\left[A_{44}w+\frac{4}{3h^2}\left((\overline{F}_{12}+2\overline{F}_{66})\,\frac{\partial^2 w}{\partial x^2}+\overline{F}_{22}\,\frac{\partial^2 w}{\partial y^2}\right)\right]\\
&\quad+(\overline{D}_{12}+\overline{D}_{66})\frac{\partial^2\psi_x}{\partial x\partial y}+\overline{D}_{66}\,\frac{\partial^2\psi_y}{\partial x^2}+\overline{D}_{22}\,\frac{\partial^2\psi_y}{\partial y^2}-\overline{A}_{44}\psi_y\\
&\quad=-\frac{4}{3h^2}\left(I_5-\frac{4}{3h^2}\,I_7\right)\frac{\partial\ddot{w}}{\partial y}+\left(I_3-\frac{8}{3h^2}\,I_5+\frac{16}{9h^4}I_7\right)\ddot{\psi}_y,
\end{aligned}\tag{19}
$$

where the modified stiffness are

$$
\begin{aligned}
&\overline{A}_{ii}=A_{ii}-\frac{8}{h^2}\,D_{ii}+\frac{16}{h^4}\,F_{ii},\quad \overline{D}_{ij}=D_{ij}-\frac{8}{3h^2}\,F_{ij}+\frac{16}{9h^4}\,H_{ij},\\
&\overline{F}_{ij}=F_{ij}-\frac{4}{3h^2}\,H_{ij}\quad (i,j=1,2,6;4,5).
\end{aligned}
$$

Eqs. (17)–(19) may be normalized using the transformation (12). The non-dimensional governing equations for free vibration analysis of symmetric honeycomb panels are written as

$$
\begin{aligned}
&\overline{a}_{55}\phi_a^2\tfrac{\partial^2 W}{\partial\xi^2}+\overline{a}_{44}\phi_b^2\tfrac{\partial^2 W}{\partial\eta^2}-\tfrac{16}{9}\left(h_{11}\phi_a^4\tfrac{\partial^4 W}{\partial\xi^4}+2(h_{12}+h_{66})\phi_a^2\phi_b^2\tfrac{\partial^4 W}{\partial\xi^2\partial\eta^2}+h_{22}\phi_b^4\tfrac{\partial^4 W}{\partial\eta^4}\right)\\
&\quad+\frac{\partial}{\partial\xi}\left[\overline{a}_{55}\phi_a\Psi_x+\tfrac{4}{3}\left(\overline{f}_{11}\phi_a^3\tfrac{\partial^2\Psi_x}{\partial\xi^2}+(\overline{f}_{12}+2\overline{f}_{66})\phi_a\phi_b^2\tfrac{\partial^2\Psi_x}{\partial\eta^2}\right)\right]\\
&\quad+\frac{\partial}{\partial\eta}\left[\overline{a}_{44}\phi_b\Psi_y+\tfrac{4}{3}\left((\overline{f}_{12}+2\overline{f}_{66})\phi_a^2\phi_b\tfrac{\partial^2\psi_y}{\partial\xi^2}+\overline{f}_{22}\phi_b^3\,\frac{\partial^2\Psi_y}{\partial\eta^2}\right)\right]\\
&\quad+\lambda^4\left[i_1W-\tfrac{16}{9}i_7\left(\phi_a^2\tfrac{\partial^2 W}{\partial\xi^2}+\phi_b^2\tfrac{\partial^2 W}{\partial\eta^2}\right)+\tfrac{4}{3}\left(i_5-\tfrac{4}{3}i_7\right)\left(\phi_a\tfrac{\partial\Psi_x}{\partial\xi}+\phi_b\frac{\partial\Psi_y}{\partial\eta}\right)\right]=0,
\end{aligned}\tag{20}
$$

$$
\begin{aligned}
&-\frac{\partial}{\partial\xi}\left[\overline{a}_{55}\phi_aW+\tfrac{4}{3h^2}\left(\overline{f}_{11}\phi_a^3\tfrac{\partial^2 W}{\partial\xi^2}+(\overline{f}_{12}+2\overline{f}_{66})\phi_a\phi_b^2\tfrac{\partial^2 W}{\partial\eta^2}\right)\right]\\
&\quad+\overline{d}_{11}\phi_a^2\tfrac{\partial^2\Psi_x}{\partial\xi^2}+\overline{d}_{66}\phi_b^2\tfrac{\partial^2\Psi_x}{\partial\eta^2}-\overline{a}_{55}\Psi_x+(\overline{d}_{12}+\overline{d}_{66})\phi_a\phi_b\tfrac{\partial^2\Psi_y}{\partial\xi\partial\eta}\\
&\quad+\lambda^4\left[-\tfrac{4}{3}\left(i_5-\tfrac{4}{3}i_7\right)\tfrac{\partial W}{\partial\xi}+\left(i_3-\tfrac{8}{3}i_5+\tfrac{16}{9}I_5\right)\Psi_x\right]=0,
\end{aligned}\tag{21}
$$

$$-\frac{\partial}{\partial\eta}\left[\overline{a}_{44}\phi_b W+\tfrac{4}{3}\left((\overline{f}_{12}+2\overline{f}_{66})\phi_a^2\phi_b\frac{\partial^2 W}{\partial\xi^2}+\overline{f}_{22}\phi_b^3\frac{\partial^2 W}{\partial\eta^2}\right)\right]$$
$$+(\overline{d}_{12}+\overline{d}_{66})\phi_a\phi_b\frac{\partial^2\psi_x}{\partial\xi\partial\eta}+\overline{d}_{66}\frac{\partial^2\Psi_y}{\partial\xi^2}+\overline{d}_{22}\frac{\partial^2\Psi_y}{\partial\eta^2}-\overline{a}_{44}\Psi_y$$
$$+\lambda^4\left[-\tfrac{4}{3}\left(i_5-\tfrac{4}{3}i_7\right)\frac{\partial W}{\partial\eta}+\left(i_3-\tfrac{8}{3}i_5+\tfrac{16}{9}i_7\right)\Psi_y\right]=0, \tag{22}$$

where

$$\overline{a}_{ii}=\frac{\overline{A}_{ij}}{E_0h}=a_{ii}-8d_{ii}+16f_{ii},\quad \overline{d}_{ij}=\frac{\overline{D}_{ij}}{E_0h^3}=d_{ij}-\tfrac{8}{3}f_{ij}+\tfrac{16}{9}h_{ij},$$

$$\overline{f}_{ij}=\tfrac{\overline{F}_{ij}}{E_0h^5}=f_{ij}-\tfrac{4}{3}h_{ij}\quad (i,j=1,2,6;4,5).$$

For a simply supported honeycomb panel, an exact analytical solution may be sought in the same series form as in Eq. (14). A standard eigenvalue problem described in Eq. (15) may be formulated upon substitution of Eq. (14) into Eqs. (20)–(22). The two matrices defining the eigenvalue problem of Eq. (15) are now written as

$$[M_{mn}]=\begin{bmatrix} i_1+\tfrac{16}{9}i_7[(m\pi\phi_a)^2+(m\pi\phi_b)^2] & -\tfrac{4}{3}\left(i_5-\tfrac{4}{3}i_7\right)m\pi\phi_a & -\tfrac{4}{3}\left(i_5-\tfrac{4}{3}i_7\right)n\pi\phi_b \\ & i_3-\tfrac{8}{3}i_5+\tfrac{16}{9}i_7 & 0 \\ \text{symmetric} & & i_3-\tfrac{8}{3}i_5+\tfrac{16}{9}i_7 \end{bmatrix},$$

$$[K_{mn}]=\begin{bmatrix} \overline{a}_{55}\overline{m}^2+\overline{a}_{44}\overline{n}^2+\varDelta_1 & a_{55}\overline{m}-\varDelta_2 & \overline{a}_{44}\overline{n}-\varDelta_3 \\ & \overline{d}_{11}\overline{m}^2+\overline{d}_{66}\overline{n}^2+\overline{a}_{55} & (\overline{d}_{12}+\overline{d}_{66})\overline{m}\,\overline{n} \\ \text{symmetric} & & \overline{d}_{66}\overline{m}^2+\overline{d}_{22}\overline{n}^2+\overline{a}_{44} \end{bmatrix},$$

where

$$\varDelta_1=\tfrac{16}{9}\{h_{11}\overline{m}^4+2(h_{12}+2h_{66})\overline{m}^2\overline{n}^2+h_{22}\overline{n}^4\},\quad \varDelta_2=\tfrac{4}{3}\{\overline{f}_{11}\overline{m}^3+(\overline{f}_{12}+\overline{f}_{66})\overline{m}\,\overline{n}^2\},$$

$$\varDelta_3=\tfrac{4}{3}\{\overline{f}_{22}\overline{n}^3+(\overline{f}_{12}+\overline{f}_{66})\overline{m}^2\overline{n}\}.$$

3. Results and discussions

For a given simply supported rectangular honeycomb panel, eigenvalues of free flexural vibration associated with each of the three plate theories may be readily obtained from the equations presented in the previous section.

As the first test case, the natural frequencies of flexural vibration for rectangular isotropic plates are studied. The plates, made of aluminum, have a length of 1.25 m, a width of 1 m, and variable thicknesses between 0.002 and 0.2 m. The material properties used in this test case are given in Table 1. The first four natural frequencies of isotropic plates, obtained using the CPT, the IPT and the TSDPT for various plate thicknesses, are plotted in Fig. 4. It is expected that the differences in

Table 1
Plate and panel descriptions

Cases	Description	Geometry	Material properties
1	Isotropic plate (three plate theories used)	$a = 1.25\,\text{m}$ $b = 1.0\,\text{m}$ h variable	$E = 69\,\text{GPa}$ $G = 26\,\text{GPa}$ $\nu = 0.33$
2	Honeycomb panel (three plate theories used)	Panel: $a = 1.25\,\text{m}, b = 1.0\,\text{m}$ Cell: $l_c = h_c = 1.833\,\text{mm}, \theta_c = 30^\circ$, $t_c = 0.0254\,\text{mm}, \alpha = h_c/h = 0.8$ two identical face sheets: $h_f = (1-\alpha)h/2$	 $E_c = 69\,\text{GPa}$ $G_c = 26\,\text{GPa}$ $\nu_c = 0.33$ $E_f = 69\,\text{GPa}$ $G_f = 26\,\text{GPa}$ $\nu_f = 0.33$
3	Honeycomb panel (TSDPT)	Panel: $a = 1\,\text{m}, b = 1.0\,\text{m}$ $h = 0.01, 0.05, 0.1, 0.2\,\text{m}$ Cell: $l_c = h_c = 1.833\,\text{mm}, \theta_c = 30^\circ$, $t_c = 0.0254\,\text{mm}$, α varying between 0 and 1 two identical face sheets: $h_f = (1-\alpha)h/2$	 $E_c = 69\,\text{GPa}$ $G_c = 26\,\text{GPa}$ $\nu_c = 0.33$ $E_f = 69\,\text{GPa}$ $G_f = 26\,\text{GPa}$ $\nu_f = 0.33$

eigenvalues or natural frequencies obtained using the CPT and the two other plate theories should be negligibly small for thin plates, and somewhat small for moderately thick, and significant for thick plates. It is known that the IPT predicts accurate natural frequencies for moderately thick isotropic plates. An examination of Fig. 4 indicates that the natural frequencies obtained using the IPT and the TSDPT are almost identical for thin, moderately thick and thick plates. However, the results obtained from the CPT are accurate only for thin plates characterized by thickness-to-length ratios (ϕ_b) less than 0.05. For moderately thick plates ($0.05 < \phi_b < 0.1$), the CPT results start to diverge from those obtained using the IPT and/or TSDPT. For thick plates ($\phi_b > 0.1$), the CPT significantly over-predicts the natural frequencies.

In the second test case, free flexural vibration of honeycomb panels is studied. The rectangular panels have two aluminum face sheets, and an aluminum core of hexagonal cells illustrated in Fig. 5. Table 1 describes the honeycomb panels and their core. The panel thicknesses are allowed to vary between 0.002 and 0.2 m. The equivalent core properties were calculated using the correlations given by Gibson and Ashby [13], and are presented here for reference: $E_1 = 0.212\,\text{MPa}, E_2 = 0.318\,\text{MPa}, E_3 = 1.4721\,\text{GPa}, G_{12} = 0.0187\,\text{MPa}, G_{13} = 208.01\,\text{MPa}, G_{23} = 329.35\,\text{MPa}, \ \rho = 57.60\,\text{kg/m}^3, \nu_{12} = 0.33, \nu_{21} = 0.4950$.

It is mentioned that the shear modulus G_{23}, used in this paper, is the average of the upper and lower bounds given by Gibson and Ashby.

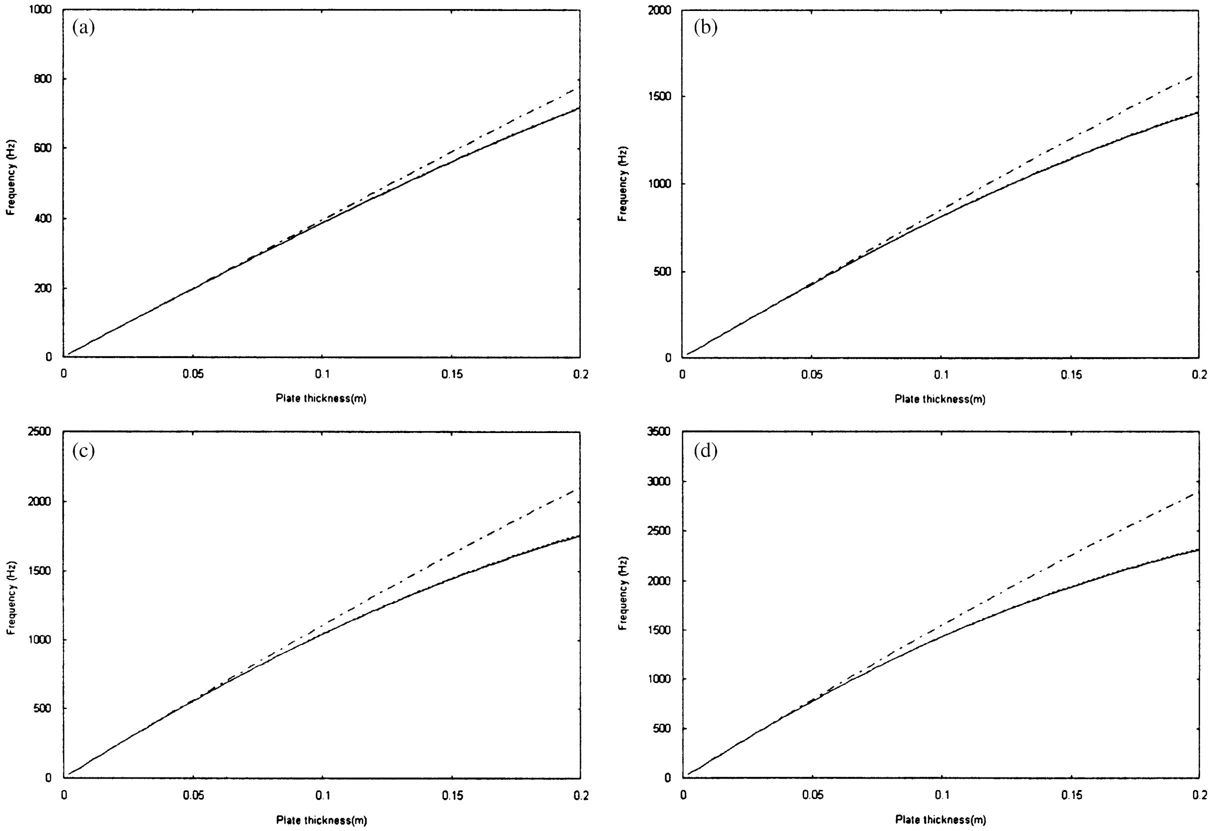


Fig. 4. (a) The first natural frequencies obtained using different plate theories for isotropic plates in case 1; (b) the second natural frequencies; (c) the third natural frequencies; (d) the fourth natural frequencies. Key: —, TSDPT; – - IPT; - - - CPT.

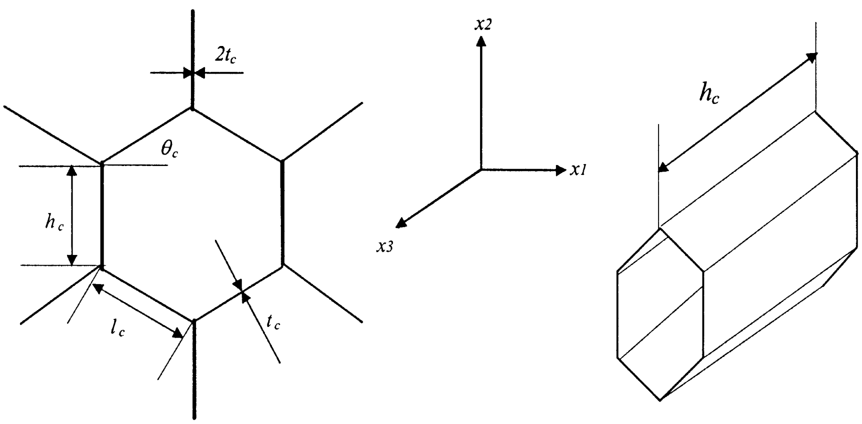


Fig. 5. Nomenclature of a typical honeycomb cell.

The first four natural frequencies versus the panel thickness, obtained using the three plate theories are plotted in Fig. 6. In this case, it is clear that the CPT and the IPT yield acceptable results only for extremely thin panels ($\phi_b < 0.01$). In general, neither the CPT nor the IPT yields satisfactory results for honeycomb panels. One of the reasons for this is that both the CPT and the IPT assume that the straight material lines normal to the mid-plane before deformation remain straight after deformation. For honeycomb panels having thick and soft cores, the straight material line assumption is no longer valid. In other words, the effects of shear deformation in honeycomb panels cannot be neglected; nor can they be adequately addressed by introducing a shear correctional coefficient. It is necessary to use a higher order plate theory, such as Reddy's, for the vibration analysis of honeycomb panels.

In the third test case, natural frequencies and frequencies normalized with respect to the panel weights were obtained using Reddy's TSDPT to study the effects of core thickness and panel thickness. For this purpose, a square honeycomb panel described in Table 1 is used. The fundamental frequencies and the frequency-to-weight ratios versus the core-to-thickness ratios are shown in Fig. 7 for four plate thickness ratios representing extremely thin, moderately thin,

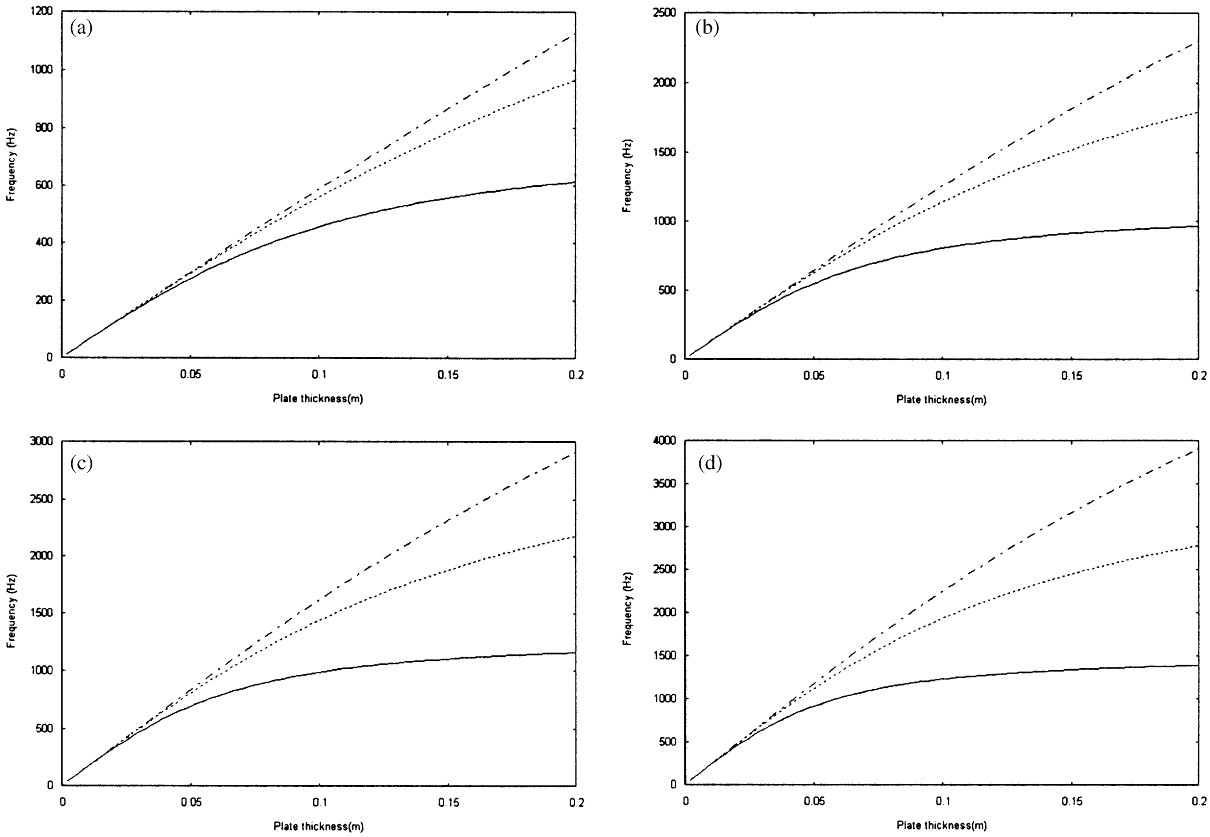


Fig. 6. (a) The first natural frequency obtained using different plate theories for honeycomb panels in case 2; (b) the second natural frequencies; (c) the third natural frequencies; (d) the fourth natural frequencies. Key: —, TSDPT; – - IPT; - - - CPT.

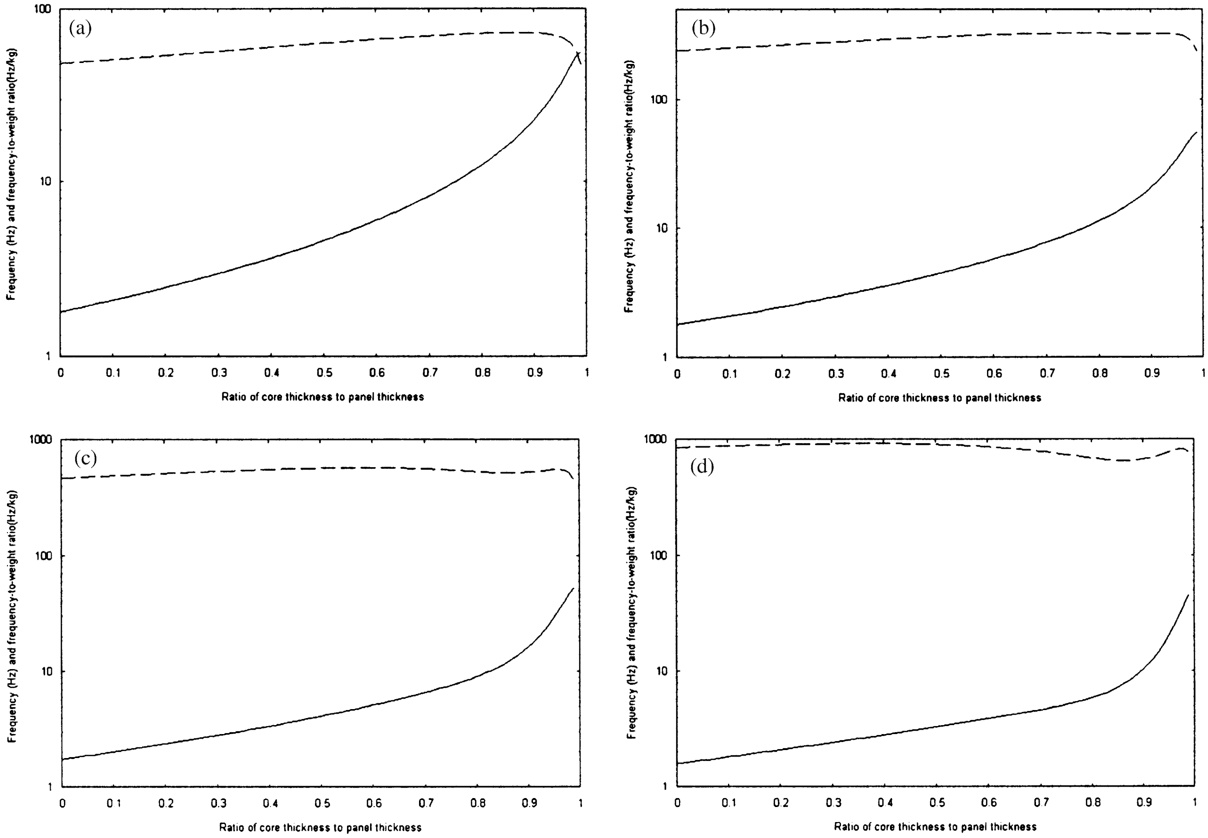


Fig. 7. The first natural frequencies and frequency-to-weight ratios vs. core-to-panel thickness ratio for: (a) an extremely thin honeycomb panel ($\phi_a = \phi_b = 0.01$) in case 3; (b) a thin honeycomb panels ($\phi_a = \phi_b = 0.05$) in case 3; (c) a moderately thick honeycomb panels ($\phi_a = \phi_b = 0.1$) in case 3; (d) an extremely thick panel ($\phi_a = \phi_b = 0.2$) in case 3. Key: —, frequencies; - - - frequency-to-weight ratios.

moderately thick, and thick panels. It can be seen that the frequency-to-weight ratios increase with the core-to-thickness ratios steadily in the range [0, 0.8], and sharply in the range [0.8, 0.98]. Many commercial grade honeycomb panels have core thickness ratios falling within the latter range. Examples include A502C, A505C, A507C and A510C of TEKLAM's aluminum honeycomb series, having core-to-panel thickness ratios of 0.8, 0.9, 0.93 and 0.95, respectively. When designing a honeycomb that has to meet the requirement of a certain absolute frequency value, it is important to realize that the natural frequencies increase with the core-to-panel thickness ratio until about 0.95, beyond which natural frequencies start to drop drastically for thin honeycomb panels. For moderately thin and thick panels, the decrease in the natural frequencies occurs at a core thickness ratio lower than 0.95, although the drastic change in natural frequencies occurs at 0.98. As a result, for thick panels, there is some penalty in achieving the absolute value of a natural frequency when designing the honeycomb panels for economic benefits. Similar conclusions, can be drawn for high vibration modes from the fourth natural frequencies, obtained for the same panels and shown in Fig. 8.

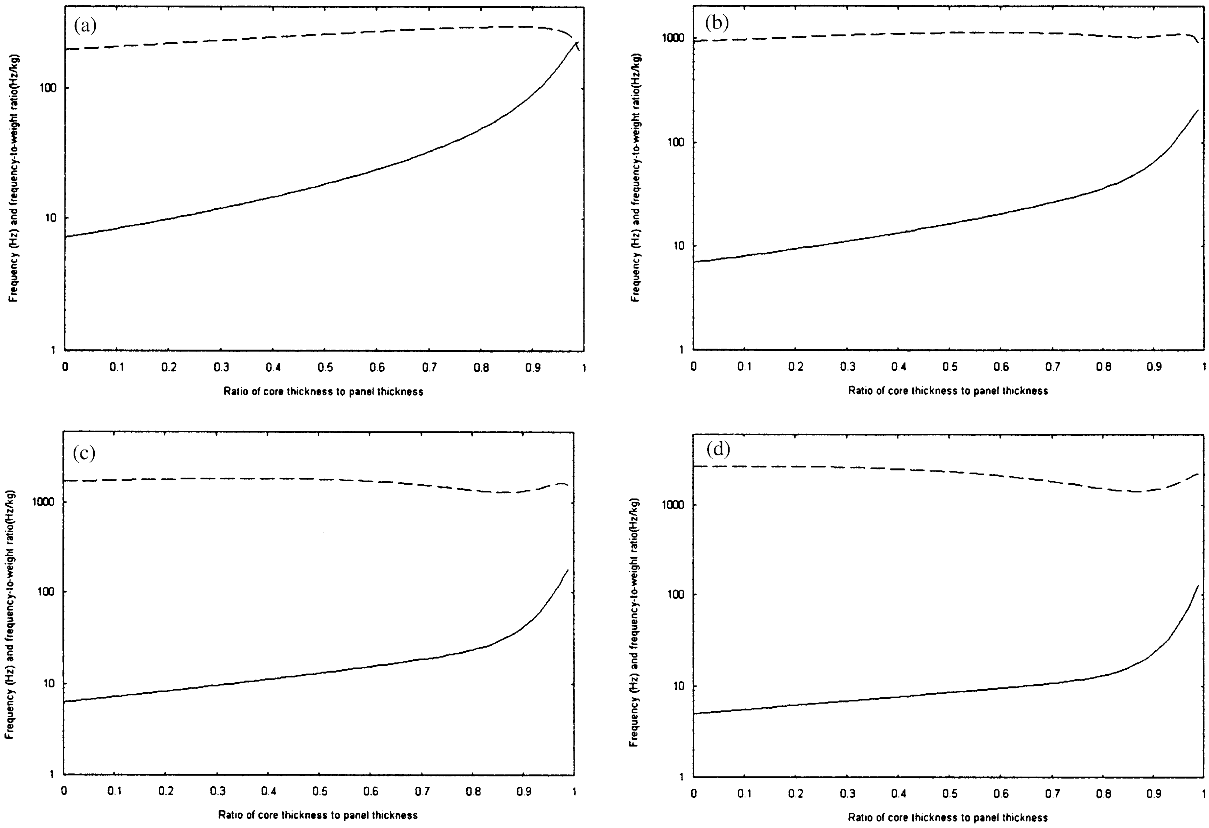


Fig. 8. The fourth natural frequencies and frequency-to-weight ratios vs. core-to-panel thickness ratio for (a) an extremely thin honeycomb panel ($\phi_a = \phi_b = 0.01$) in case 3; (b) a thin honeycomb panels ($\phi_a = \phi_b = 0.5$) in case 3; (c) a moderately thick honeycomb panels ($\phi_a = \phi_b = 0.1$) in case 3; (d) an extremely thick panel ($\phi_a = \phi_b = 0.2$) in case 3. Key: —, frequencies; - - - frequency-to-weight ratios.

4. Conclusions

This paper presents comparative studies of the free flexural vibration of honeycomb panels using three different plate theories. Numerical results indicate that the classical and improved plate theories are not adequate for the flexural vibration analysis of honeycomb panels. Using the Reddy third order shear deformation plate theory and Gibson and Ashby correlations, the authors investigated the effects of panel thickness and core thickness on flexural vibration of symmetric honeycomb panels.

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